Classical Coding Problem from Transversal T Gates

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Joint Work: Michael Newman, Robert Calderbank and Henry Pfister

2020 International Symposium on Information Theory (ISIT '20)

arXiv:2001.04887, 1910.09333, 1902.04022, 1907.00310

June 21-26, 2020

Overview

Goal: Logical Operations from Physical Gates

Information $|x\rangle_L$

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Information $|x\rangle_L \xrightarrow{\text{logical operation}} |\tilde{x}\rangle_L$

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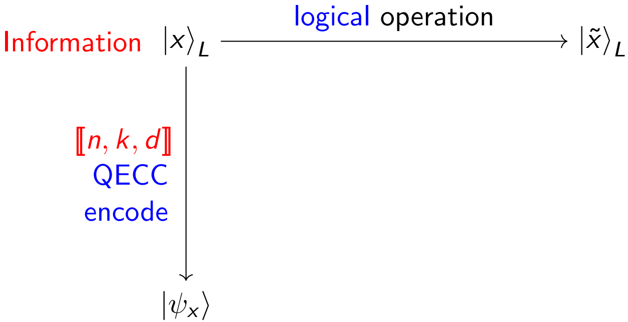
QECC: Quantum Error Correcting Code



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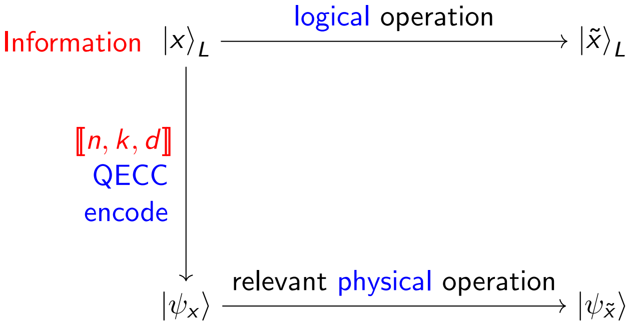

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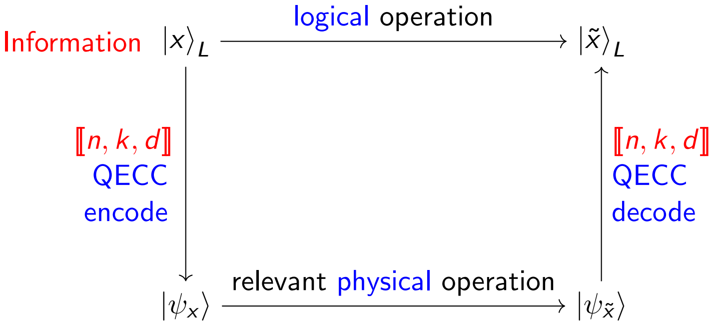

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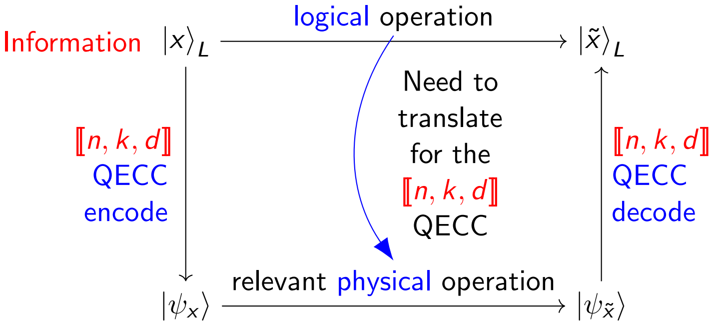

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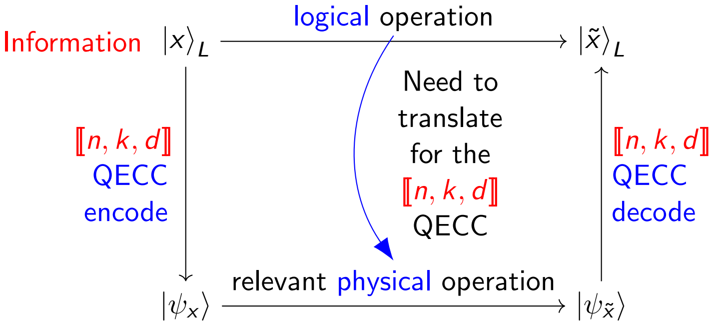

What QECC structure is required so that
the physical application of certain gates preserves the code subspace?

Line of Thought

What QECC structure is required so that
the physical application of certain gates preserves the code subspace?

Key Idea

Pauli operators form an orthonormal basis for all operators!

- Understand action of those certain gates on Pauli operators
- Use the action to study effect on QECC subspaces
- Finally, restrict to gates that are reliable in the lab

Literature related to Magic State Distillation (MSD)

- [GC99]: Universal computation via quantum teleportation
- [BK05]: Ideal Clifford gates and noisy ancillas – MSD
- [BH12]: Distillation with low overhead, triorthogonal codes
- [KB15]: Transversal gates on color codes
- [CH17]: Quasitransversality
- [HH17]: Generalized triorthogonality
- [Haa+17]: Distillation with optimal asymptotic input count
- [KT18]: Punctured polar codes from decreasing monomial codes
- [VB19]: Quantum Pin Codes
- . . . (see arXiv:1910.09333 for explicit connections)

Main Distinction of Our Work

Prior works ("Schrödinger Perspective"):

- Focus on Calderbank-Shor-Steane (CSS) type stabilizer codes
- Examine action of the (physical) gates on the basis quantum states in the CSS code subspace

Our strategy ("Heisenberg Perspective"):

- Work with arbitrary stabilizer codes; results can be specialized to CSS
- Examine action of the (physical) gates on the Pauli operators defining the code subspace

In this talk ...

1. Motivation and Related Work
2. Essential Algebraic Setup
3. Quadratic Form Diagonal (QFD) Gates
4. Stabilizer Codes Matched to QFD Gates

Overview

Pauli Group, Clifford Group and Symplectic Matrices

Heisenberg-Weyl Group $HW_N := \{\imath^{\kappa} E(a,b) \colon a, b \in \mathbb{F}_2^n, \kappa \in \mathbb{Z}_4\}$ $(\imath = \sqrt{-1})$

$E(a,b), a, b \in \mathbb{F}_2^n$: $\underbrace{X \otimes Z \otimes Y}_{n=3 \text{ qubits}} = E(\underbrace{1\,0\,1}_{a}, \underbrace{0\,1\,1}_{b})$

$a =$	1	0	1
$b =$	0	1	1
$E(a,b) =$	X_1	Z_2	Y_3

Symplectic Inner Product: $\langle [a,b], [c,d] \rangle_s := [a,b]\,\Omega\,[c,d]^T, \Omega := \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$

Clifford Group: All unitaries that map Paulis to Paulis under conjugation

Symplectic Matrices: If $g \in \text{Cliff}_N$ (Cliffords on $n = \log_2 N$ qubits) then

$$g\, E(a,b)\, g^{\dagger} = \pm E\left([a,b]F_g\right), \text{ where } F_g \Omega F_g^T = \Omega$$

$F_g \in \mathbb{F}_2^{2n \times 2n}$ is symplectic: preserves the symplectic inner product

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Stabilizer Codes ($N = 2^n$)

r-dimensional Stabilizer: Generated by r commuting Pauli operators:

$$S = \langle \epsilon_i E(a_i, b_i)\, ;\, i = 1, \ldots, r \rangle,\ \epsilon_i \in \{\pm 1\},\ -I_N \notin S$$

$[\![n, k = n - r, d]\!]$ Stabilizer Code: The 2^k dimensional subspace, $V(S)$, jointly fixed by all elements of S

$$V(S) := \left\{ |\psi\rangle \in \mathbb{C}^N : g\,|\psi\rangle = |\psi\rangle \ \text{for all } g \in S \right\}$$

Example:

$[\![6, 4, 2]\!]$ CSS Code: $S := \langle X^{\otimes 6} = E(a, 0), Z^{\otimes 6} = E(0, a) \rangle$, $a := [\,1\,1\,1\,1\,1\,1\,]$

Generator Matrix: $G_S = \left[\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Overview

Gates for Universal Computation

$\text{Cliff}_N = \langle H, P, \text{CZ or CNOT (on all qubits)} \rangle \longleftarrow$ Not universal!

Gate	Unitary Matrix	Action on Paulis	Symplectic Matrix
Hadamard	$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	$HXH^\dagger = Z$ $HZH^\dagger = X$	$F_H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Phase	$P := \begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix} = \sqrt{Z}$	$PXP^\dagger = Y$ $PZP^\dagger = Z$	$F_P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
Phase (P), Ctrl-Z (CZ)	$t_R := \sum_{v \in \mathbb{F}_2^n} \imath^{vRv^T} \lvert v \rangle \langle v \rvert$ (vRv^T computed over $\mathbb{Z}$)	CZ: $X_a \mapsto X_a Z_b$ $Z_a \mapsto Z_a$	$T_R = \begin{bmatrix} I_n & R \\ 0 & I_n \end{bmatrix}$ with R symmetric
T	$T := \begin{bmatrix} 1 & 0 \\ 0 & e^{\imath\pi/4} \end{bmatrix} = \sqrt{P}$	$TXT^\dagger = \frac{X+Y}{\sqrt{2}}$ $TZT^\dagger = Z$	?

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Quadratic Form Diagonal (QFD) Gates

S.X. Cui, D. Gottesman and A. Krishna, Phys. Rev. A, 2017
If $U \in \mathcal{C}^{(\ell)}$ is diagonal, then all entries are 2^{ℓ}-th roots of unity.

Examples:

$P \in \mathcal{C}^{(2)} \leftrightarrow R = [\,1\,]$ over $\mathbb{Z}_4$

$\mathcal{C}^{(2)}: \; t_R = \sum_{v \in \mathbb{F}_2^n} \imath^{vRv^T} |v\rangle \langle v|$

R is $n \times n$ symmetric with entries in $\mathbb{Z}_2$

$\mathcal{C}^{(1)} = HW_N$

$$\text{CZ} = \text{diag}\,[1, 1, 1, -1] \in \mathcal{C}^{(2)}$$

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$\mathcal{C}^{(\ell)}: \tau_R^{(\ell)} = \sum_{v \in \mathbb{F}_2^n} \xi^{vRv^T} |v\rangle \langle v|$

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$\text{CP} = \text{diag}\,[1, 1, 1, \imath] \in \mathcal{C}^{(3)}$

$\leftrightarrow R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{Z}_8$

Diagonal Recursion for QFD Gates

Recollect: Clifford g acts as $g\, E(a,b)\, g^\dagger = \pm E([a,b]F_g)$, F_g symplectic.

How do QFD gates act on Pauli matrices under conjugation?

$$\tau_R^{(\ell)} E(a,b) \left(\tau_R^{(\ell)}\right)^\dagger = \phi(R,a,b,\ell) \cdot E\left([a,b]\begin{bmatrix} I_n & R \\ 0 & I_n \end{bmatrix}\right) \cdot \tau_{\tilde{R}(R,a,\ell)}^{(\ell-1)}$$

$\phi(R,a,b,\ell)$: Deterministic global phase

$\tilde{R}(R,a,\ell)$: New symmetric matrix with entries in $\mathbb{Z}_{2^{\ell-1}}$

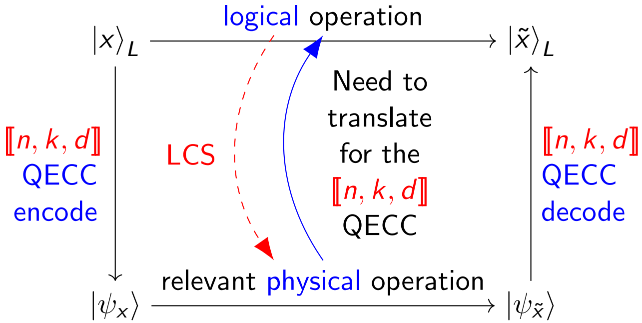
All 1- and 2-local diagonal gates in $\mathcal{C}^{(\ell)}$ are QFD for any $\ell \geq 1$

For details see: https://arxiv.org/abs/1902.04022

Overview

Reverse Strategy for Physical T Gates

QECC: Quantum Error Correcting Code
LCS: Logical Clifford Synthesis (arXiv:1907.00310)



What stabilizer structure is required so that
the physical application of T gates preserves the code subspace?

Transversal T as a Logical Operator

Question: When is transversal T a logical operator for a stabilizer code? What is the induced logical operation?

Stabilizer: $S = \langle \epsilon_i E(a_i, b_i) \, ; \, i = 1, 2, \ldots, r \rangle, \; \epsilon_i \in \{\pm 1\}$

Code Projector: $\Pi_s = \prod_{i=1}^{r} \dfrac{I_N + \epsilon_i E(a_i, b_i)}{2} = \dfrac{1}{2^r} \sum_{a,b \in S} \epsilon_{a,b} E(a, b)$

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Calculation using QFD recursion [hard for general QFD!]

$$T^{\otimes n} E(a, b) \left(T^{\otimes n}\right)^{\dagger} = \frac{1}{2^{\mathrm{wt}_H(a)/2}} \sum_{y \preceq a} (-1)^{b y^T} E(a, b \oplus y)$$

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$T^{\otimes n}$ is a logical operator iff $T^{\otimes n} \Pi_S (T^{\otimes n})^{\dagger} = \Pi_S$: [also hard in general!]

$$\frac{1}{2^r} \sum_{a,b \in S} \frac{\epsilon_{a,b}}{2^{\mathrm{wt}_H(a)/2}} \sum_{y \preceq a} (-1)^{b y^T} E(a, b \oplus y) = \frac{1}{2^r} \sum_{a,b \in S} \epsilon_{a,b} E(a, b)$$

CSS-T Codes and Two Corollaries

CSS-T Codes: Pair (C_1, C_2) of codes satisfying $C_2 \subset C_1$ and the following:

1. All codewords $x \in C_2$ have even Hamming weight $w_H(x)$.
2. For each $x \in C_2$, $C_1^{\perp}$ consists of a dimension $w_H(x)/2$ self-dual code Z_x supported on x (i.e., Z_x is essentially a $[w_H(x), w_H(x)/2]$ code).

This yields a quantum code with parameters $[\![n, k_1 - k_2, d \geq \min(d_1, d_2^{\perp})]\!]$.

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Two Corollaries: (Non-degenerate $\Rightarrow$ each stabilizer has weight $\geq d$)

1. Triorthogonal codes [BH12]: only CSS family with $T^{\otimes n} \equiv \bar{T}^{\otimes k}$.
2. For each $[\![n, k, d]\!]$ non-degenerate stabilizer code that supports transversal T, there is an $[\![n, k, d]\!]$ CSS-T code that does too.

Classical Coding Problem

CSS-T Codes: Pair (C_1, C_2) of codes satisfying $C_2 \subset C_1$ and the following:

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Open Problem

A CSS-T family with $\dfrac{(k_1 - k_2)}{n} = \Omega(1)$ and $\dfrac{d}{n} = \Omega(1)$

Would imply constant overhead magic state distillation! $\left[\gamma = \frac{\log(n/k)}{\log d}\right]$

(see arXiv:1910.09333, or arXiv:2001.04887 for shorter version)

Summary and Future Work

- Characterized QFD gates in the Clifford hierarchy
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  - Paper: Extensions to finer angle Z-rotations and Reed-Muller codes
- Use our recipe to find codes supporting any reliable QFD gate?

Key Takeaway

Expressing unitaries in the Pauli basis seems like an under-utilized trick

References

[GC99] Daniel Gottesman and Isaac L. Chuang. "Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations". In: *Nature* 402.6760 (1999), pp. 390–393. URL: http://www.nature.com/articles/46503.

[BK05] Sergey Bravyi and Alexei Kitaev. "Universal quantum computation with ideal Clifford gates and noisy ancillas". In: *Phys. Rev. A* 71.2 (2005), p. 022316. URL: https://arxiv.org/abs/quant-ph/0403025.

[BH12] Sergey Bravyi and Jeongwan Haah. "Magic-state distillation with low overhead". In: *Phys. Rev. A* 86.5 (2012), p. 052329. URL: http://arxiv.org/abs/1209.2426.

References

[KB15] Aleksander Kubica and Michael E. Beverland. "Universal transversal gates with color codes: A simplified approach". In: *Phys. Rev. A* 91.3 (2015), p. 032330. DOI: 10.1103/PhysRevA.91.032330. URL: https://arxiv.org/abs/1410.0069.

[CH17] Earl T Campbell and Mark Howard. "Unified framework for magic state distillation and multiqubit gate synthesis with reduced resource cost". In: *Phys. Rev. A* 95.2 (Feb. 2017), p. 022316. DOI: 10.1103/PhysRevA.95.022316. URL: https://journals.aps.org/pra/pdf/10.1103/PhysRevA.95.022316.

References

[HH17] Jeongwan Haah and Matthew B. Hastings. "Codes and Protocols for Distilling T, controlled-S, and Toffoli Gates". In: *Quantum* 2 (2017), p. 71. DOI: 10.22331/q-2018-06-07-71. URL: https://arxiv.org/abs/1709.02832.

[Haa+17] Jeongwan Haah et al. "Magic state distillation with low space overhead and optimal asymptotic input count". In: *Quantum* 1 (2017), p. 31. DOI: 10.22331/q-2017-10-03-31. URL: http://arxiv.org/abs/1703.07847.

[KT18] Anirudh Krishna and Jean-Pierre Tillich. "Magic state distillation with punctured polar codes". In: *arXiv preprint arXiv:1811.03112* (2018). URL: http://arxiv.org/abs/1811.03112.

References

[RCP19] Narayanan Rengaswamy, Robert Calderbank, and Henry D. Pfister. "Unifying the Clifford Hierarchy via Symmetric Matrices over Rings". In: *Phys. Rev. A* 100.2 (2019), p. 022304. DOI: 10.1103/PhysRevA.100.022304. URL: http://arxiv.org/abs/1902.04022.

[Ren+19a] Narayanan Rengaswamy et al. "Logical Clifford Synthesis for Stabilizer Codes". In: *arXiv preprint arXiv:1907.00310* (2019). URL: http://arxiv.org/abs/1907.00310.

[Ren+19b] Narayanan Rengaswamy et al. "On Optimality of CSS Codes for Transversal T". In: *arXiv preprint arXiv:1910.09333* (2019). URL: http://arxiv.org/abs/1910.09333.

[VB19] Christophe Vuillot and Nikolas P. Breuckmann. "Quantum Pin Codes". In: *arXiv preprint arXiv:1906.11394* (2019). URL: http://arxiv.org/abs/1906.11394.

Thank you!

For details see: http://arxiv.org/abs/1910.09333

QFD Gates: http://arxiv.org/abs/1902.04022

LCS Algorithm: http://arxiv.org/abs/1907.00310
Code at https://github.com/nrenga/symplectic-arxiv18a

Any feedback is much appreciated.