On Cyclic Polar Codes and the Burst Erasure Performance of Spatially-Coupled LDPC Codes

Narayanan Rengaswamy
Graduate Student in Electrical Engineering
Texas A&M University

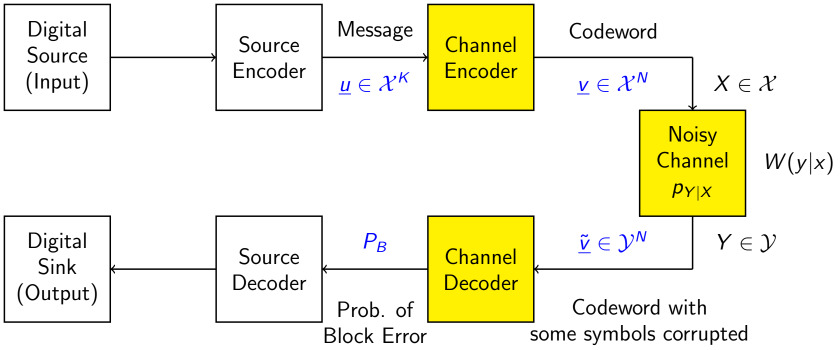
Advisor: Dr. Henry D. Pfister

Master's Thesis Defense

Oct 8, 2015

Overview

Communication System: Focus of the Thesis



Shannon's model of a communication system [Sha48].

Focus is on the channel coding blocks.

Fundamentals of Coding Theory: Goal

- Capacity: maximum rate of transmission for reliable communication over a given channel.

$$\boxed{C \triangleq \max_{p_X} I(X;Y)} \quad \text{(symbols/channel use)}$$

- Capacity-achieving codes: A sequence of codes, indexed by n, with

$$\text{Code Rate, } \boxed{R_n = \frac{K_n}{N_n}} \text{ s.t. } \boxed{\lim_{n\to\infty} R_n = C \text{ with } P_{B_n}^{\max} \to 0}$$

$\Rightarrow$ fundamental limit on code rate, given the channel.

- Construction of capacity-achieving codes – goal for over 60 years!

Fundamentals of Coding Theory: Goal

- Capacity: maximum rate of transmission for reliable communication over a given channel.

$$\boxed{C \triangleq \max_{p_X} I(X;Y)} \quad \text{(symbols/channel use)}$$

- Capacity-achieving codes: A sequence of codes, indexed by n, with

$$\text{Code Rate, } \boxed{R_n = \frac{K_n}{N_n}} \text{ s.t. } \boxed{\lim_{n\to\infty} R_n = C \text{ with } P_{B_n}^{\max} \to 0}$$

$\Rightarrow$ fundamental limit on code rate, given the channel.

- Construction of capacity-achieving codes – goal for over 60 years!
- Research culminated in Polar Codes [Arı09] and Spatially-Coupled Low-Density Parity-Check Codes [FZ99; KRU11].

Overview

Polar Codes: Introduction

- Introduced by Arıkan in [Arı09] using the binary 2×2 kernel

$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Longrightarrow \text{Transform: } \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_0 & u_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} u_0 + u_1 \\ u_1 \end{bmatrix}.$$

- Length $N = 2^n$ binary polar transform matrix is given by

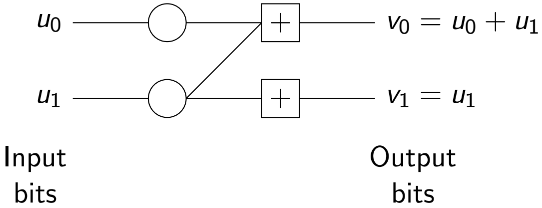
$$\boxed{G_N = B_N G_2^{\otimes n}} \Longrightarrow \text{Transform: } \boxed{\underline{v} = \underline{u}^T G_N},$$

where B_N is the length-N bit-reversal permutation matrix and $G_2^{\otimes n} = \underbrace{G_2 \otimes G_2 \otimes \cdots \otimes G_2}_{n \text{ times}}$ denotes the Kronecker product of G_2 performed n times.

- Explicitly shown to achieve the *symmetric capacity* of binary input DMCs, asymptotically, under successive cancellation (SC) decoding.

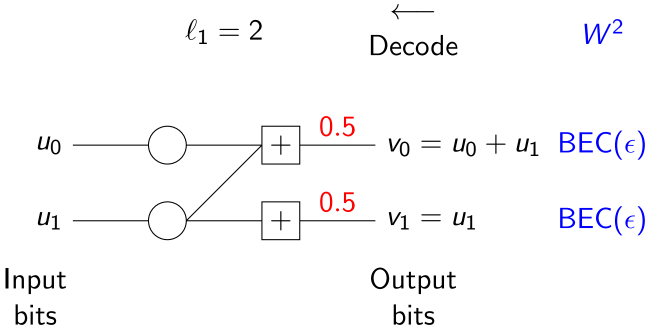
Polar Codes: Polarization on the BEC

$$\ell_1 = 2$$



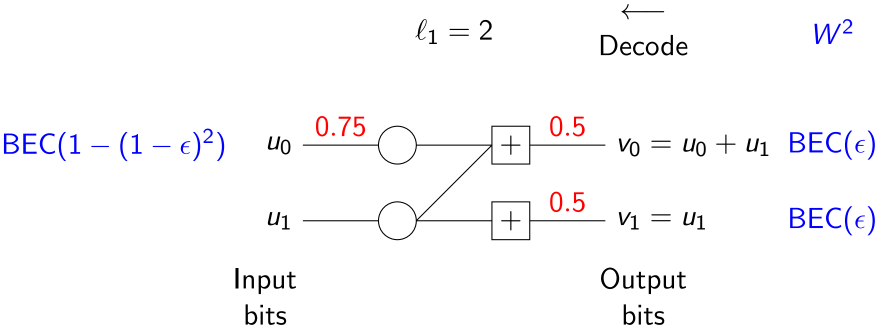
Density Evolution (DE) for Binary Polar Code with $N = 2$.

Polar Codes: Polarization on the BEC



Density Evolution (DE) for Binary Polar Code with $N = 2$.

Polar Codes: Polarization on the BEC



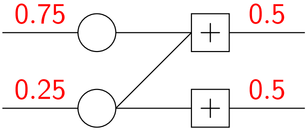
Density Evolution (DE) for Binary Polar Code with $N = 2$.

Polar Codes: Polarization on the BEC

$W_2^{(i)}$ $(i = 0, 1)$ | $\ell_1 = 2$ | $\longleftarrow$ Decode | W^2

$\text{BEC}(1-(1-\epsilon)^2)$ u_0

$\text{BEC}(\epsilon^2)$ u_1



$v_0 = u_0 + u_1$ $\text{BEC}(\epsilon)$

$v_1 = u_1$ $\text{BEC}(\epsilon)$

Input bits

Output bits

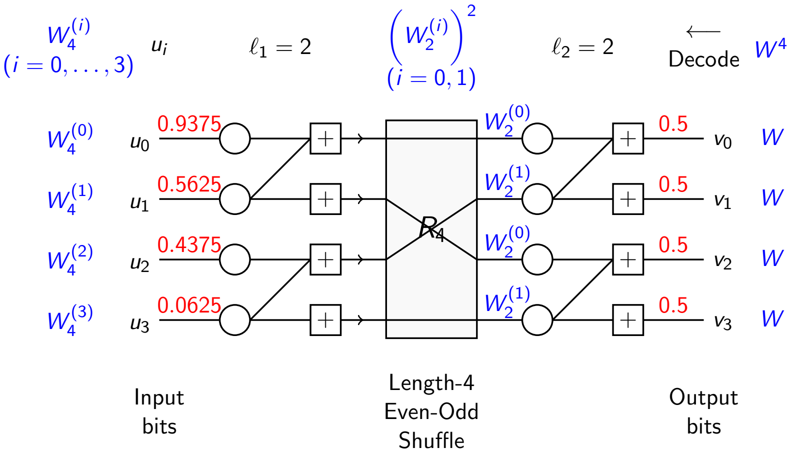
Coordinate Channels $\Rightarrow$

$$W_2^{(0)} : u_0 \to (v_0, v_1)$$

$$W_2^{(1)} : u_1 \to (v_0, v_1, u_0)$$

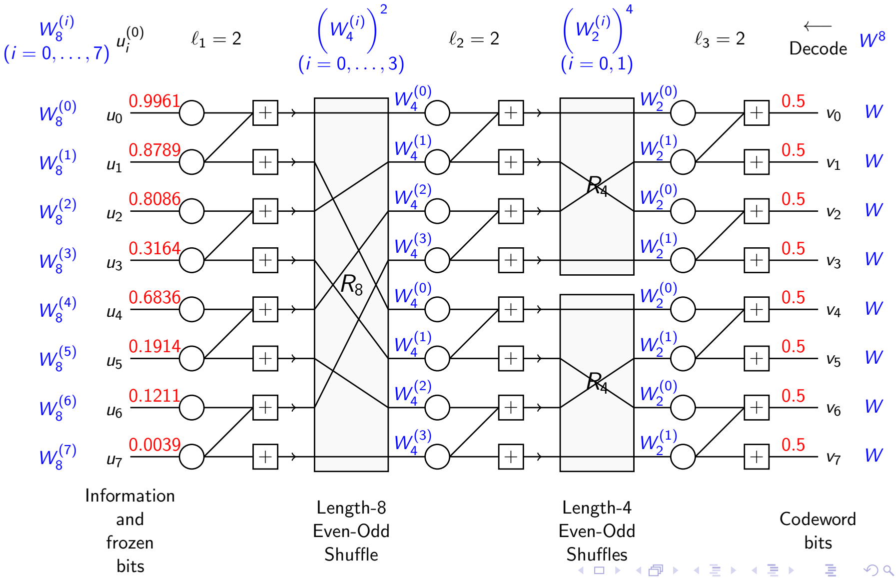
Density Evolution (DE) for Binary Polar Code with $N = 2$.

Polar Codes: Polarization on the BEC



Density Evolution (DE) for Binary Polar Code with $N = 2 \cdot 2 = 4$.

Polar Codes: Polarization on the BEC



Polar Codes: Capacity-Achieving on B-DMCs

Main Theorem of Arıkan [Arı09, Theorem 1]

For any B-DMC W, the **channels $\{W_N^{(i)}\}$ polarize** so that, for any $0 < \theta < 1$, as $N = 2^n \to \infty$ we have

$$\frac{1}{N}\left|\left\{i : I(W_N^{(i)}) \in (1-\theta, 1]\right\}\right| = I(W).$$

i.e. **the fraction of "good" channels is equal to the symmetric capacity**, $I(W)$, of the underlying channel W.

Proof.

Martingale convergence analysis of rate and reliability random processes associated to a channel tree process. □

Polar Codes: Finite Blocklength Rates

Binary Polar Code – Rates for BEC($\epsilon = 0.5$), $P_B \leq \delta = 0.1$ (DE)

Blocklength N	**Rate R**
$2^3 = 8$	0.125
$2^4 = 16$	0.25
$2^6 = 64$	0.2812
$2^7 = 128$	0.3125
$2^8 = 256$	0.3281
$2^{16} = 65536$	0.4397

Capacity $= 1 - \epsilon = 0.5$.

Rates approach capacity slowly.

Polar Codes: Extensions

Blocklength $N = \ell^n$, $\ell > 2$, use an $\ell \times \ell$ kernel G_ℓ.

Transformation: $G_N = B_N G_\ell^{\otimes n}$

- Korada et al.: Binary G_ℓ for binary DMCs [KŞU10].
- Şaşoğlu et al.: Binary G_ℓ for q-ary DMCs, q prime [ŞTA09].

Polar Codes: Extensions

Blocklength $N = \ell^n$, $\ell > 2$, use an $\ell \times \ell$ kernel G_ℓ.

Transformation: $G_N = B_N G_\ell^{\otimes n}$

- Korada et al.: Binary G_ℓ for binary DMCs [KŞU10].
- Şaşoğlu et al.: Binary G_ℓ for q-ary DMCs, q prime [ŞTA09].

- Mori and Tanaka: Non-binary G_ℓ for arbitrary symmetric q-ary DMCs, $q = p^m$ [MT10; MT14].
  - For example, using extended RS matrices.

$$G_{RS}(3,3) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^0 & \alpha^0 & 0 \\ \alpha^1 & \alpha^0 & 0 \\ \alpha^0 & \alpha^0 & \alpha^0 \end{bmatrix}$$

where $\alpha = 2 \in \mathbb{F}_3$ is primitive.

Cyclic Polar Codes: Motivation

So far, blocklength $N = \ell^n$ and transformation $G_N = B_N G_\ell^{\otimes n}$.

How about mixed-size kernels?

Cyclic Polar Codes: Motivation

So far, blocklength $N = \ell^n$ and transformation $G_N = B_N G_\ell^{\otimes n}$.

How about mixed-size kernels?

At finite blocklengths, can we achieve higher rates than binary polar codes?

Cyclic Polar Codes: Motivation

System implementation: Many systems use RS or other cyclic codes.

Polar codes are closely related to Reed-Muller (RM) codes.

Can we relate polar codes to Reed-Solomon (RS) codes?

Cyclic Polar Codes: Motivation

System implementation: Many systems use RS or other cyclic codes.

Polar codes are closely related to Reed-Muller (RM) codes.

Can we relate polar codes to Reed-Solomon (RS) codes?

More fundamentally, can we make polar codes cyclic?

Overview

Galois Field Fourier Transform (GFFT)

- Replace polar transform with Galois field Fourier Transform (GFFT).
- Input: $\underline{u} = (u_0, u_1, \ldots, u_{N-1})$, Output: $\underline{v} = (v_0, v_1, \ldots, v_{N-1})$. Then

$$\boxed{\underline{u} \xrightarrow{GIFFT} \underline{v}}$$

- If F_N is the GFFT matrix,

$$\underline{u} = F_N \underline{v} \qquad \text{(or)} \qquad u_j = \sum_{i=0}^{N-1} v_i \omega^{ij}$$

for $j = 0, 1, \ldots, N-1$, where ω in $\mathbb{F}_q$ has order N.

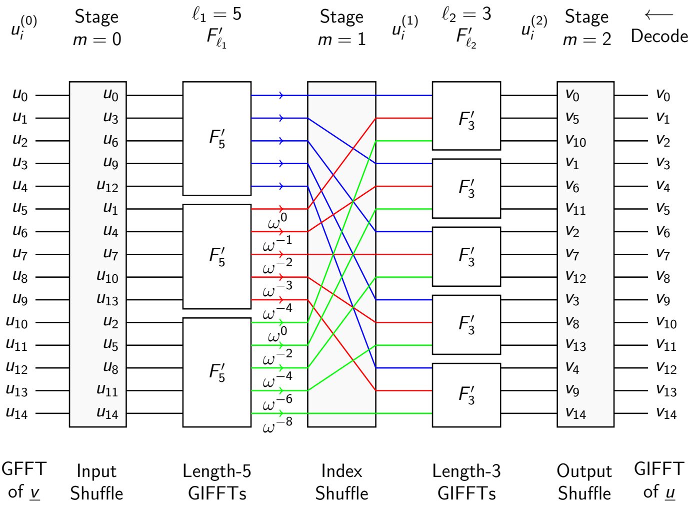
Galois Field Fourier Transform (GFFT)

- How can this transform be implemented efficiently?

Galois Field Fourier Transform (GFFT)

- How can this transform be implemented efficiently?
- Evaluate the GFFT using the Cooley-Tukey FFT [CT65].
  - Factor $N = \prod_{m=1}^{n} \ell_m = \ell_1 \ell_2 \cdots \ell_n$.
  - Implement small GFFTs of length ℓ_m directly.
  - Combine them using appropriate twiddle factors and index-shuffles.
  - Simplest case is $\ell_1 = \ell_2 = \cdots = \ell_n = 2$ for $N = 2^n$; equivalent to standard polar code over a larger field.
- Ignoring twiddle factors, the Kronecker product of repeated short GFFTs gives a long GFFT.

GFFT: Example (two stages)



An example for $N = 5 \cdot 3 = 15$ over $\mathbb{F}_{16}$ depicting the transform.

GFFT: Kronecker Product Formulation

Lemma: Length-ab GFFT

$$F_{ab} = S_{b,a}(I_a \otimes F_b)D_{a,b}(F_a \otimes I_b),$$

where I_a: $a \times a$ identity matrix, $[D_{a,b}]_{i,i} = \omega_{ab}^{\lfloor i/b \rfloor (i \bmod b)}$.

$$\text{E.g. } F_{15} = S_{3,5}(I_5 \otimes F_3)D_{5,3}(F_5 \otimes I_3).$$

Lemma: Length-$N = \prod_{m=1}^{n} \ell_m$ GFFT

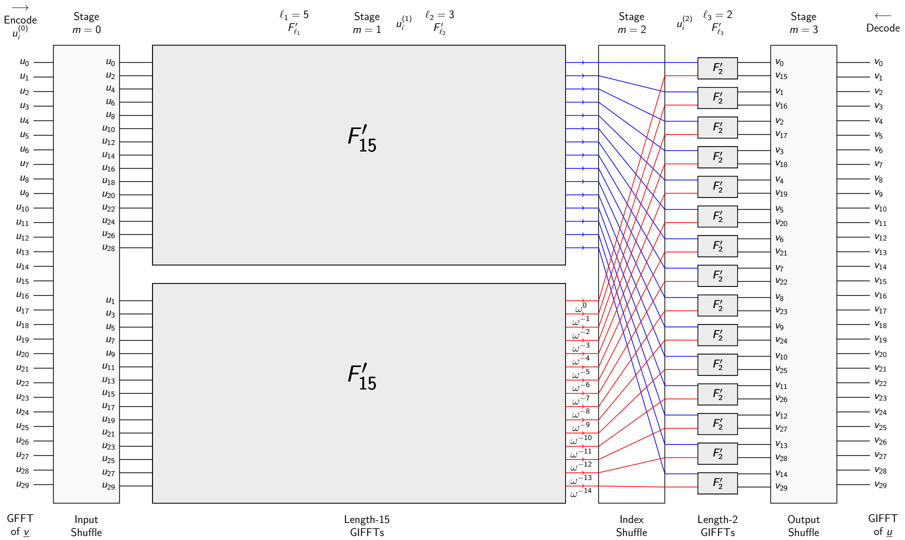
Let $p_j = \prod_{m=1}^{j} \ell_m$. Then

$$F_N = U_n U_{n-1} \cdots U_1,$$

where

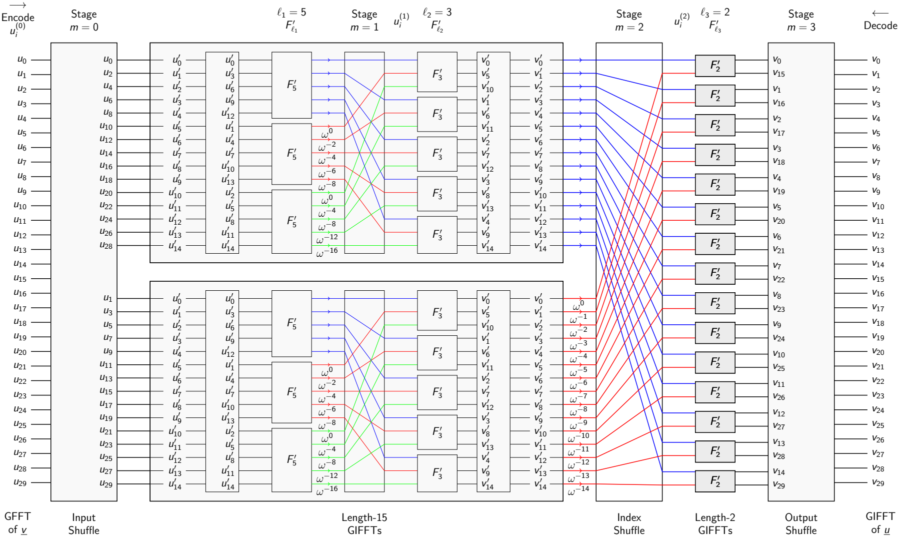
$$U_m = (S_{N/p_m,\ell_m} D_{\ell_m,N/p_m} \otimes I_{p_m/\ell_m})(F_{\ell_m} \otimes I_{N/\ell_m}).$$

GFFT: Example (three stages)



An example for $N = 15 \cdot 2 = 30$ over $\mathbb{F}_{31}$ depicting the transform.

GFFT: Example (three stages)



An example for $N = 5 \cdot 3 \cdot 2 = 30$ over $\mathbb{F}_{31}$ depicting the transform.

Cyclic Polar Codes (CP Codes)

- Recollect that

$$u_j = \sum_{i=0}^{N-1} v_i \omega^{ij}$$

where $\omega^N = 1$ in $\mathbb{F}_q$.

- In polynomial notation, with $v(x) = \sum_{i=0}^{N-1} v_i x^i$, we have

$$u(x) = \sum_{j=0}^{N-1} \boldsymbol{u_j} x^j = \sum_{j=0}^{N-1} \boldsymbol{v(\omega^j)} x^j$$

- Hence, u_j's are evaluations of $v(x)$.

Cyclic Polar Codes (CP Codes)

- Design of the code $\mathcal{C}$ produces the set of information indices $\mathcal{A}$.
- Given $\mathcal{A}^c$, the set of indices frozen to zeros in $u(x)$ such that $u_j = 0 \; \forall \, j \in \mathcal{A}^c$, there exists a generator $g(x)$ such that

$$v(x) = u_{\mathcal{A}}(x) g(x) = u_{\mathcal{A}}(x) \prod_{j \in \mathcal{A}^c} (x - \omega^j)$$

where $\omega^N = 1$ in $\mathbb{F}_q$.

- We have a cyclic code!

Cyclic Polar Codes (CP Codes)

- Design of the code $\mathcal{C}$ produces the set of information indices $\mathcal{A}$.
- Given $\boldsymbol{\mathcal{A}^c}$, the set of indices frozen to zeros in $\boldsymbol{u(x)}$ such that $\boldsymbol{u_j = 0 \; \forall \; j \in \mathcal{A}^c}$, there exists a generator $g(x)$ such that

$$v(x) = u_{\mathcal{A}}(x)\boldsymbol{g(x)} = u_{\mathcal{A}}(x) \prod_{\boldsymbol{j \in \mathcal{A}^c}} \boldsymbol{(x - \omega^j)}$$

  where $\omega^N = 1$ in $\mathbb{F}_q$.
- We have a cyclic code!
- Constraint: $N|(q-1)$. Hence, field size must grow with the blocklength.

Cyclic Polar Codes (CP Codes)

Is this transformation polarizing?

Cyclic Polar Codes (CP Codes)

- Example: $N = 3$ and $N = 5$ over $\mathbb{F}_{16}$.

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \quad \text{and} \quad F_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

  where $\omega^3 = 1$ for F_3, $\omega^5 = 1$ for F_5.

- The transformation F_N, a GFFT matrix, polarizes any q-ary channel because it
  - is invertible and not upper triangular.
  - contains a primitive element [MT14].

Overview

Soft or Hard Decoder?

- 2×2 kernel:
  - Compute optimal soft estimates for inputs from outputs using standard techniques from LDPC codes.
  - Hence, use optimal soft-decision decoding.
- $\ell \times \ell$ kernel, $\ell > 2$: (APP: A Posteriori Probability)
  - Hard to implement APP decoder for general length-ℓ code over $\mathbb{F}_q$.
  - Hence, use algebraic hard-decision decoding.

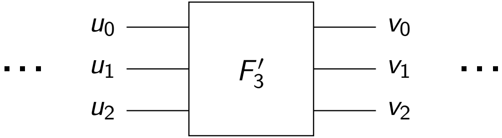
CP Codes: Design for Algebraic Erasures Decoding

q-ary Erasure Channel (QEC) with $\epsilon = 0.5$.

Use Forney's algebraic decoder.

$$u_i \in \mathbb{F}_{16},\ v_i \in \mathbb{F}_{16} \cup \{?\}$$

$u_i^{(m-1)}$ $\quad \ell_m = 3 \quad$ $u_i^{(m)}$



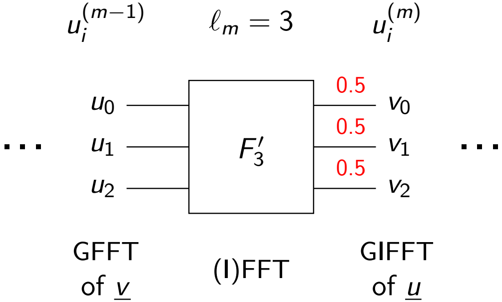
GFFT of $\underline{v}$ $\quad$ (I)FFT $\quad$ GIFFT of $\underline{u}$

Uncover next (unknown) input using outputs and recovered inputs.

CP Codes: Design for Algebraic Erasures Decoding

q-ary Erasure Channel (QEC) with $\epsilon = 0.5$.
Use Forney's algebraic decoder.

$$u_i \in \mathbb{F}_{16},\ v_i \in \mathbb{F}_{16} \cup \{?\},\ \epsilon = P\{v_i = ?\} = 0.5$$



Uncover next (unknown) input using outputs and recovered inputs.

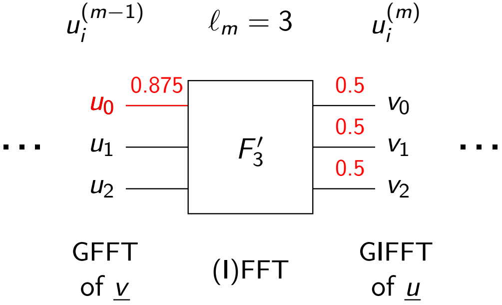
CP Codes: Design for Algebraic Erasures Decoding

q-ary Erasure Channel (QEC) with $\epsilon = 0.5$.
Use Forney's algebraic decoder.

$$u_i \in \mathbb{F}_{16},\ v_i \in \mathbb{F}_{16} \cup \{?\},\ \epsilon = P\{v_i = ?\} = 0.5$$

Theoretical

$1 - (1-\epsilon)^3$



Uncover next (unknown) input using outputs and recovered inputs.

CP Codes: Design for Algebraic Erasures Decoding

q-ary Erasure Channel (QEC) with $\epsilon = 0.5$.

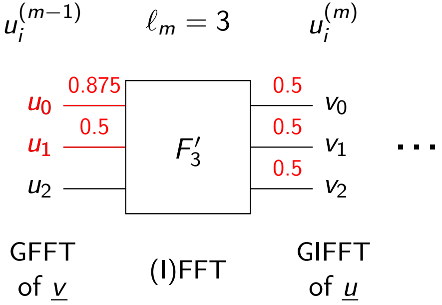
Use Forney's algebraic decoder.

$$u_i \in \mathbb{F}_{16},\ v_i \in \mathbb{F}_{16} \cup \{?\},\ \epsilon = P\{v_i = ?\} = 0.5$$

Theoretical

$1 - (1-\epsilon)^3$

$1 - [3(1-\epsilon)^2\epsilon + (1-\epsilon)^3]$



Uncover next (unknown) input using outputs and recovered inputs.

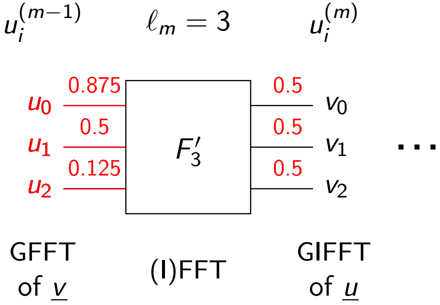
CP Codes: Design for Algebraic Erasures Decoding

q-ary Erasure Channel (QEC) with $\epsilon = 0.5$.
Use Forney's algebraic decoder.

$$u_i \in \mathbb{F}_{16},\ v_i \in \mathbb{F}_{16} \cup \{?\},\ \epsilon = P\{v_i = ?\} = 0.5$$

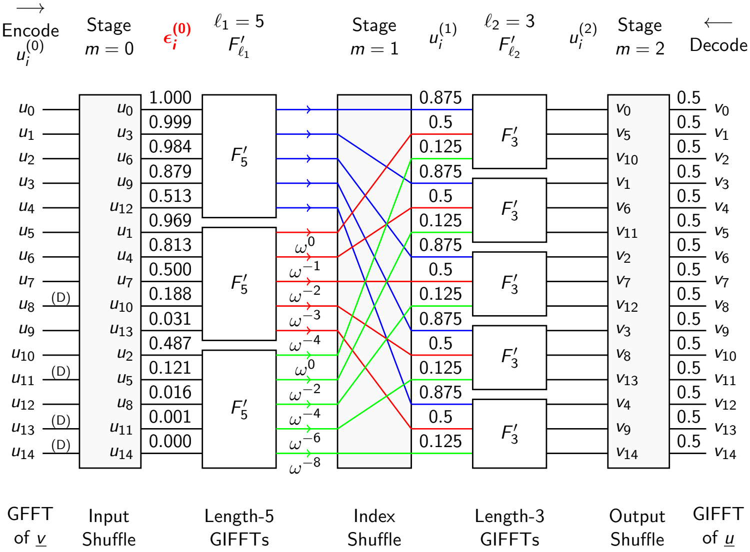
Theoretical

$$1 - (1-\epsilon)^3$$
$$1 - [3(1-\epsilon)^2\epsilon + (1-\epsilon)^3]$$
$$\epsilon^3$$



Uncover next (unknown) input using outputs and recovered inputs.

CP Codes: Subcodes of higher rate RS codes



CP Codes: One Stage of Polarization

Erasure decoding of a GFFT block of length ℓ: $(j = 0, 1, \ldots, \ell - 1)$

$$\psi(\ell, j, \epsilon') \triangleq \sum_{i=0}^{(\ell-1)-j} \binom{\ell}{i} (1-\epsilon')^i (\epsilon')^{\ell-i}.$$

Lemma

Above equation: $\mathbb{R} \rightarrow \mathbb{R}^\ell$ with properties:

(*i*) The mapping preserves the mean erasure rate:

$$\frac{1}{\ell} \sum_{j=0}^{\ell-1} \psi(\ell, j, \epsilon') = \epsilon'.$$

(*ii*) If $\epsilon' \in (0, 1)$, then new channels polarize:

$$\psi(\ell, \ell-1, \epsilon') < \epsilon' < \psi(\ell, 0, \epsilon').$$

CP Codes: Capacity-Achieving on QEC

Main Theorem for Cyclic Polar Codes

For any QEC W with capacity $(1-\epsilon)$, the **channels $\{W_N^{(i)}\}$ polarize** so that, for any $0 < \theta < 1$, as $N \to \infty$ we have

$$\frac{1}{N}\left|\left\{ i : \epsilon_i^{(0)} \in [0, \theta) \right\}\right| = 1 - \epsilon.$$

i.e. **the fraction of "good" channels is equal to the symmetric capacity**, $(1-\epsilon)$, of the underlying channel.

Proof.

Use previous lemma in Arıkan's martingale convergence analysis with appropriate changes. □

CP Codes: Finite Blocklength Rates

Cyclic Polar Code – Rates for QEC($\epsilon = 0.5$), $P_B \leq \delta = 0.1$ (DE)

Blocklength *N*	**Rate *R***
$2^3 = 8$	0.125
12	0.25
13	0.3077
14	0.2857
$2^4 = 16$	0.25
30	0.2667, 0.3
60	0.2833, 0.3, 0.3167
$2^6 = 64$	0.2812
255	0.3843, 0.3882, 0.3922, 0.3961
$2^8 = 256$	0.3281
1023	0.4291, 0.4340
$2^{16} = 65536$	0.4397

Code Design for Algebraic Errors and Erasures Decoding

- For erasures-only case, we used Forney's algorithm to decode.
- If errors are also present:
  - Use Berlekamp-Massey (B-M) algorithm to locate errors.
  - Once error positions are known, use Forney's algorithm to correct them.
- Problem: The B-M algorithm can mislocate errors when $\nu > t$, where ν: actual # of errors & erasures and $2t$: # of inputs known.
- Possible solution based on numerical results: Work in suitable larger fields, $\mathbb{F}_q$, and pass erasures back when B-M detects $\nu > t$.
- The design using DE should consider the error-correction capability, i.e. at least two inputs need to be known to correct even one error in the output.

Decoding Complexity

- For a length-ℓ block, bounded by $C\ell^2$ operations for some $C > 0$.
- Since there are $\prod_{j\neq m} \ell_j = N/\ell_m$ blocks at stage ℓ_m, the decoding complexity is bounded by

$$\sum_{m=1}^{n} \prod_{j\neq m} \ell_j \left(C\ell_m^2\right) = CN \sum_{m=1}^{n} \ell_m \leq CNn \max_m \ell_m.$$

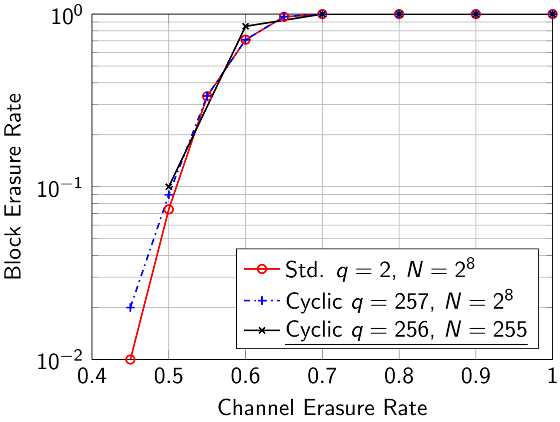
- For comparison, complexity of standard polar codes for $N = 2^n$ is $O(N \log N)$ under SC decoding.

Performance of Cyclic Polar Codes

What is the performance of cyclic polar codes?

Overview

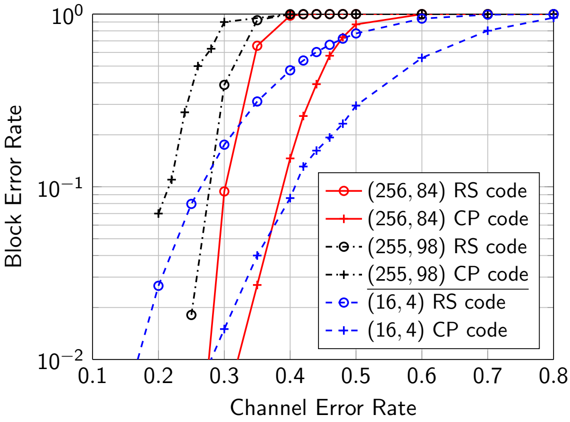
Simulation Results on q-ary Erasure Channels (QEC)



Comparison of performance of standard polar and cyclic polar codes.

$\delta = 0.1$; $\epsilon = 0.5$ produced rates 0.328 for $N = 256$, 0.384 for $N = 255$.

Simulation Results on q-ary Symmetric Channels (QSC)



Performance of QEC-designed cyclic polar (CP) codes on the QSC.

$\delta = 0.1$; $\epsilon = 0.5$ produced rates 0.328 for $N = 256$, 0.384 for $N = 255$, 0.25 for $N = 16$.

Overview

Summary

- Cyclic Polar (CP) code construction for arbitrary blocklength N that explicitly achieve the symmetric capacity of QEC.
- These codes are subcodes of higher rate RS codes; an RS decoder can be used with suboptimal performance.

Summary

- Cyclic Polar (CP) code construction for arbitrary blocklength N that explicitly achieve the symmetric capacity of QEC.
- These codes are subcodes of higher rate RS codes; an RS decoder can be used with suboptimal performance.
- Performance of CP code with soft and hard decoding:
  - QEC: Outperforms standard polar code under deterministic hard-decision SC decoding with much higher rates and larger polarization kernels; demonstrated for $N = 255 = 17 \cdot 5 \cdot 3$.

Summary

- Cyclic Polar (CP) code construction for arbitrary blocklength N that explicitly achieve the symmetric capacity of QEC.
- These codes are subcodes of higher rate RS codes; an RS decoder can be used with suboptimal performance.
- Performance of CP code with soft and hard decoding:
  - QEC: Outperforms standard polar code under deterministic hard-decision SC decoding with much higher rates and larger polarization kernels; demonstrated for $N = 255 = 17 \cdot 5 \cdot 3$.
  - QSC: With the design on QEC, outperforms hard-decision decoding of a similar RS code under soft-decision SC decoding with much higher rates; demonstrated for $N = 256 = 2^8$.

Summary

- Cyclic Polar (CP) code construction for arbitrary blocklength N that explicitly achieve the symmetric capacity of QEC.
- These codes are subcodes of higher rate RS codes; an RS decoder can be used with suboptimal performance.
- Performance of CP code with soft and hard decoding:
  - QEC: Outperforms standard polar code under deterministic hard-decision SC decoding with much higher rates and larger polarization kernels; demonstrated for $N = 255 = 17 \cdot 5 \cdot 3$.
  - QSC: With the design on QEC, outperforms hard-decision decoding of a similar RS code under soft-decision SC decoding with much higher rates; demonstrated for $N = 256 = 2^8$.
  - QSC: With the design on QEC, inferior to hard-decision decoding of a similar RS code under algebraic hard-decision SC decoding; demonstrated for $N = 255 = 17 \cdot 5 \cdot 3$.

ISIT 2015

This work was presented in the
2015 IEEE International Symposium on Information Theory (ISIT 2015).

N. Rengaswamy and H. D. Pfister, "Cyclic Polar Codes," in *Proc. IEEE Int. Symp. Inform. Theory*, Jun. 2015, pp. 1287–1291. DOI: 10.1109/ISIT.2015.7282663

Overview

Summer Internship at Bell Labs

This work was performed during my summer internship at

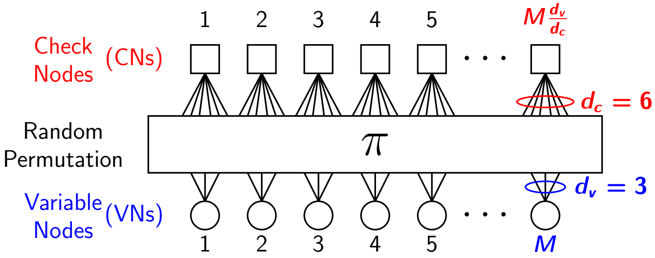
Alcatel-Lucent Bell Labs, Stuttgart, Germany,

under the supervision of
Dr. Laurent Schmalen and Dr. Vahid Aref.

Work submitted to the 2016 International Zürich Seminar (IZS 2016).

Low-Density Parity-Check (LDPC) Codes

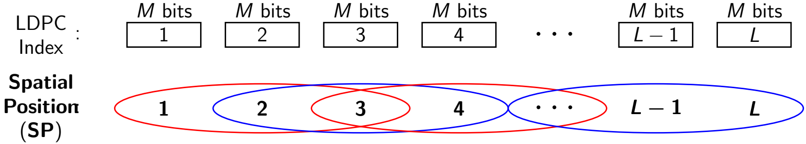
Single LDPC Code: (d_v, d_c) **Regular Ensemble**



Spatially-Coupled LDPC Codes

Spatially-Coupled LDPC Code: (d_v, d_c, w, L, M) **Random Regular Ensemble**

Blocklength: $\boxed{N = LM}$



Coupling Parameter: $w = 3$

Edge Randomization: An edge of a VN at SP i can randomly connect to any of the wMd_v edges from the $wM\frac{d_v}{d_c}$ CNs in SPs $i, i+1, \ldots, i+w-1$.

Overview

Single Position Burst Channel (SPBC)

- Erases exactly one spatial position (SP).
- Other bits are undisturbed, i.e. no random erasures.
- Practical scenario - Slotted-ALOHA:
  - Each user sends bits of one SP of a SC-LDPC codeword in one time slot

    $\Rightarrow L$ transmissions required per codeword.
  - If exactly one collision in the L time slots, we have SPBC scenario.

On the Single Position Burst Channel (SPBC)

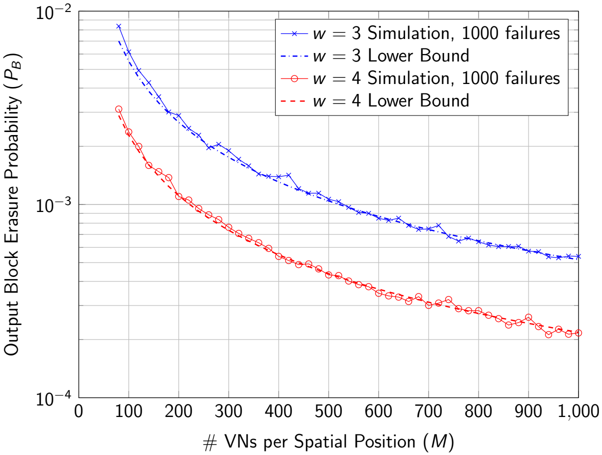
- P_B^{SPBC}: Block Erasure Probability on the SPBC or Probability of decoder failure.
- Lower bound on P_B^{SPBC}:

$$\begin{aligned} P_B^{SPBC} &= \text{Prob}\ [\text{At least one stopping set in a SP}] \\ &\geq \text{Prob}\ [\mathbb{N}_2^{SP} \geq 1] \\ &= 1 - \text{Prob}\ [\mathbb{N}_2^{SP} = 0] \\ &= 1 - e^{-\lambda_{SP}} \\ &= 1 - e^{-\binom{M}{2}p}, \end{aligned}$$

where p: probability of finding a size-2 stopping set in a SP and $\mathbb{N}_2^{SP} \sim \text{Poisson}(\lambda_{SP})$.

Simulations on SPBC

Monte Carlo simulations on the SPBC with a $(3,6)$ random ensemble.

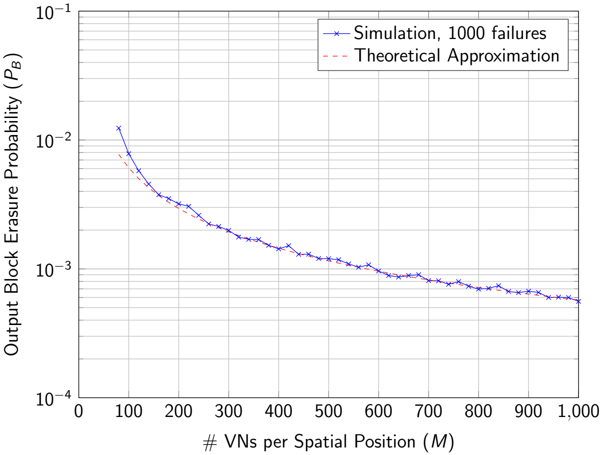


Random Burst Channel (RBC)

- Erases one set of consecutive bits in a codeword.
- Other bits are undisturbed, i.e. no random erasures.
- Represented as RBC(s, b); $b \in \{0, 1, \ldots, n\}$: Length of the burst.
- $s \in \{1, \ldots, M\}$: starting bit index of burst.
  - Represents offset from first bit of the first SP affected by the burst.
  - All SPs are structured identically $\Rightarrow$ SP index does not matter.
- Practical Scenario - Block Fading Channel.

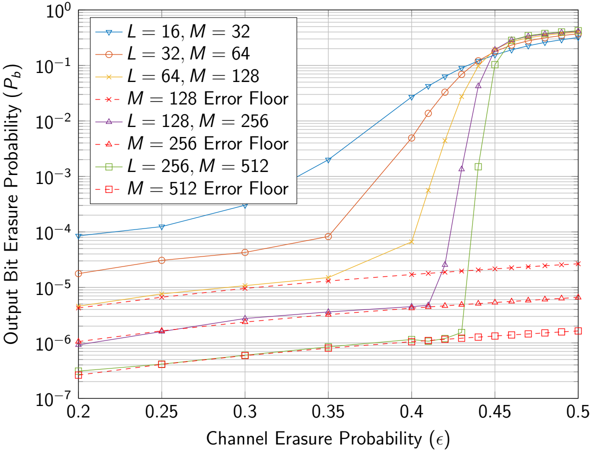
Simulations on RBC

Simulations for a $(3, 6, 3, 20, M)$ random ensemble on the RBC$(b = 1.25M)$



Error Floor on BEC

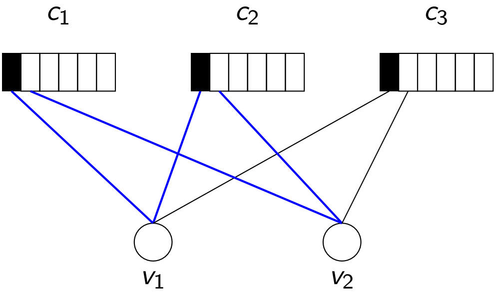
Expected error floor for a $(3, 6, 3, L, M)$ random ensemble on BEC



Overview

What is Expurgation?

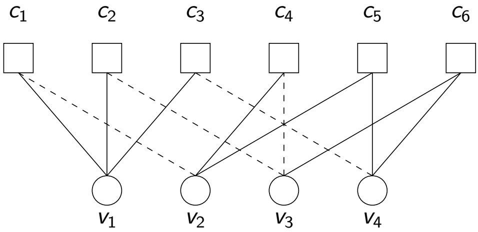
- 4-cycles usually present in the code.



- **Expurgation** - Removal of all 4-cycles in the code.

What is Expurgation?

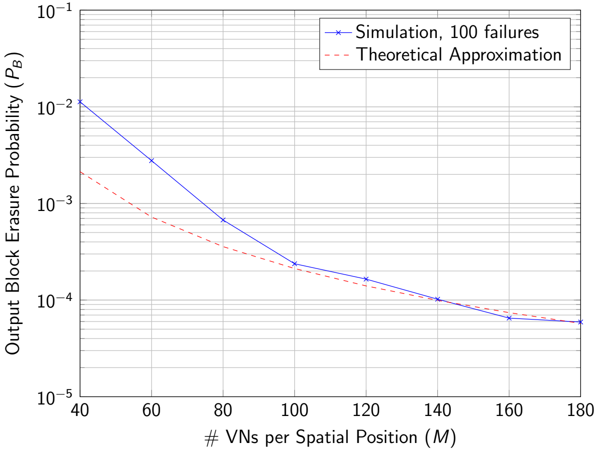
- No 4-cycles; minimum is a 6-cycle now.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Simulations on SPBC

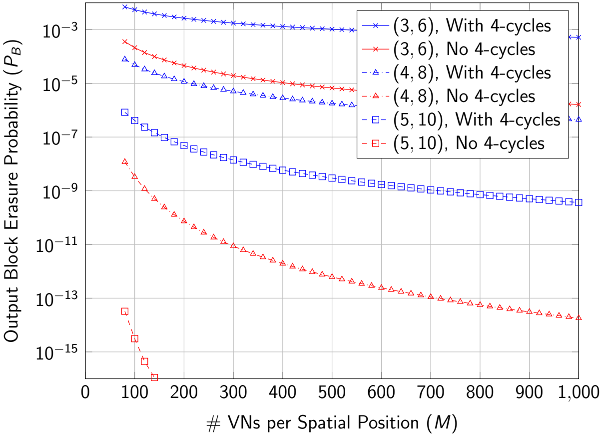
Monte Carlo simulations on the SPBC with a $(3, 6)$ expurgated random ensemble.



Overview

Comparison of Ensembles on SPBC

The lower bound on P_B^{SPBC} for various ensembles



References

[1] C. Shannon, "A mathematical theory of communication," *The Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, Jul. 1948, ISSN: 0005-8580. DOI: 10.1002/j.1538-7305.1948.tb01338.x.

[2] E. Arıkan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. Inform. Theory*, vol. 55, no. 7, pp. 3051–3073, Jul. 2009.

[3] J. Felstrom and K. S. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," *IEEE Trans. Inform. Theory*, vol. 45, no. 6, pp. 2181–2191, Sep. 1999, ISSN: 0018-9448.

References

[4] S. Kudekar, T. J. Richardson, and R. L. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Trans. Inform. Theory*, vol. 57, no. 2, pp. 803–834, Feb. 2011.

[5] S. B. Korada, E. Şaşoğlu, and R. Urbanke, "Polar codes: Characterization of exponent, bounds, and constructions," *IEEE Trans. Inform. Theory*, vol. 56, no. 12, pp. 6253–6264, 2010.

[6] E. Şaşoğlu, I. Telatar, and E. Arıkan, "Polarization for arbitrary discrete memoryless channels," in *Proc. IEEE Inform. Theory Workshop*, Oct. 2009, pp. 144–148.

[7] R. Mori and T. Tanaka, "Non-binary polar codes using Reed-Solomon codes and algebraic geometry codes," in *Proc. IEEE Inform. Theory Workshop*, Aug. 2010, pp. 1–5.

References

[8] R. Mori and T. Tanaka, "Source and channel polarization over finite fields and Reed-Solomon matrices," *IEEE Trans. Inform. Theory*, vol. 60, no. 5, pp. 2720–2736, 2014.

[9] J. Cooley and J. Tukey, "An algorithm for the machine calculation of complex Fourier series," *Mathematics of Computation*, vol. 19, no. 90, pp. 297–301, 1965.

[10] N. Rengaswamy and H. D. Pfister, "Cyclic Polar Codes," in *Proc. IEEE Int. Symp. Inform. Theory*, Jun. 2015, pp. 1287–1291. DOI: 10.1109/ISIT.2015.7282663.

Thank you!

Questions?